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NONLINEAR SINGULAR STURM-LIOUVILLE PROBLEMS WITH IMPULSIVE CONDITIONS

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Abstract. In this paper, we consider a non-linear impulsive Sturm-Liouville problem on semiinfinite intervals in which the limit-circle case holds at infinity for THE Sturm-Liouville expression. We prove the existence and uniqueness theorems for this problem.

Keywords: Impulsive Sturm-Liouville problem; Singular point; Weyl limit-circle case; Completely continuous operator; Fixed point theorems.

1. Introduction

The theory of differential equations with impulses describes processes that are subjected to abrupt changes in their states at certain moments. Such processes arise in many fields of science and technology: chemical technology, biotechnology, theoretical physics, industrial robotics, etc. For an introduction to the basic theory of differential equations with impulses see Bainov and Simeonov ([3], [4], [5]), Benchohra, Henderson and Ntouyas ([6]), Lakshmikantham, Bainov and Simeonov ([18]) Samoilenko and Perestyuk ([31]) and the references therein.

Recently, much work has been done on the existence of solutions to impulsive Sturm-Liouville equations; for regular impulsive Sturm-Liouville problems see [2, 7, 9, 12-15, 25-27, 30, 33], for singular impulsive Sturm-Liouville equations see [1, 10, 18-19, 21-24, 29]. However, there is no paper concerned with the existence of solutions to singular impulsive non-linear Sturm-Liouville problems that the limit-circle case holds at infinity. In this paper, we fill the gap by using a special way to pose boundary conditions at infinity.

Let us consider the following nonlinear Sturm-Liouville equation

$$(1.1) \quad l(y) := -(p(x)y')' + q(x)y = f(x, y), \quad x \in I,$$

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where $I := I_1 \cup I_2$, $I_1 := [a, c)$, $I_2 := (c, +\infty)$, $-\infty < a < c < +\infty$, and $y = y(x)$ is a desired solution.

Let $L^2(I)$ be a Hilbert space which is composed of all complex-valued functions y satisfying

$$\int_a^\infty |y(x)|^2 dx < \infty$$

in relation to the inner product

$$(y, z) := \int_a^\infty y(x) \overline{z(x)} dx.$$

Denote by \mathcal{D} the linear set of all functions $y \in L^2(I)$ such that y, py' are locally absolutely continuous functions on I , one-sided limits $y(c\pm), (py')(c\pm)$ exist and are finite and $l(y) \in L^2(I)$. The operator L defined by $Ly = l(y)$ is called the maximal operator on $L^2(I)$.

For two arbitrary functions $y, z \in \mathcal{D}$, we have Green's formula

$$(1.2) \quad \int_a^\infty l(y) \overline{z} dx - \int_a^\infty y \overline{l(z)} dx = [y, z]_{c-} - [y, z]_a + [y, z]_\infty - [y, z]_{c+},$$

where $[y, z]_x = y(x) \overline{(pz')(x)} - (py')(x) \overline{z(x)}$ ($x \in I$).

We assume that the following conditions are satisfied.

(A1) The points a and c are regular for the differential expression l . p and q are real-valued, Lebesgue measurable functions on I and $\frac{1}{p}, q \in L^1_{loc}(I)$. The point c is regular if $\frac{1}{p}, q \in L^1[c - \epsilon, c + \epsilon]$ for some $\epsilon > 0$. Moreover, the functions p and q are such that all solutions of the the equation

$$(1.3) \quad l(y) = 0$$

belong to $L^2(I)$, i.e., Weyl limit-circle case holds for the differential expression l (see [1-3]).

(A2) The function $f(x, y)$ is real-valued and continuous in $(x, \zeta) \in I \times \mathbb{R}$, and

$$(1.4) \quad |f(x, \zeta)| \leq g(x) + \vartheta |\zeta|$$

for all $(x, \zeta) \in I \times \mathbb{R}$, where $g(x) \geq 0$, $g \in L^2(I)$, and ϑ is a positive constant.

If we define the operator F taking each function $y(\cdot)$ to the function $f(\cdot, y(\cdot))$, then the condition (4) is necessary and sufficient for F to map $L^2(I)$ into itself (see ([17], Chapter 1)).

Denote by

$$u := u(x) = \begin{cases} u^{(1)}(x), & x \in I_1 \\ u^{(2)}(x), & x \in I_2 \end{cases}, \quad v := v(x) = \begin{cases} v^{(1)}(x), & x \in I_1 \\ v^{(2)}(x), & x \in I_2 \end{cases}$$

the solutions to the equation (1.3) satisfying the initial conditions

$$(1.5) \quad u^{(1)}(a) = 0, \quad (pu^{(1)'})'(a) = 1, \quad v^{(1)}(a) = -1, \quad (pv^{(1)'})'(a) = 0,$$

and impulsive conditions

$$(1.6) \quad \begin{aligned} U(c+) &= CU(c-), \quad U(x) := \begin{pmatrix} u(x) \\ (pu')(x) \end{pmatrix}, \\ V(c+) &= CV(c-), \quad V(x) := \begin{pmatrix} v(x) \\ (pv')(x) \end{pmatrix}, \\ C &\in M_2(\mathbb{R}), \quad \det C = \rho > 0, \end{aligned}$$

where $M_2(\mathbb{R})$ denotes the 2×2 matrices with entries from \mathbb{R} .

Now, we introduce the Hilbert space $H = L^2(I_1) \dot{+} L^2(I_2)$ with the inner product

$$\langle y, z \rangle := \int_a^c y^{(1)} \overline{z^{(1)}} dx + \gamma \int_c^\infty y^{(2)} \overline{z^{(2)}} dx, \quad \gamma = \frac{1}{\rho},$$

where

$$y(x) = \begin{cases} y^{(1)}(x), & x \in I_1 \\ y^{(2)}(x), & x \in I_2 \end{cases}, \quad z(x) = \begin{cases} z^{(1)}(x), & x \in I_1 \\ z^{(2)}(x), & x \in I_2. \end{cases}$$

We set $W_x^{(i)} := W_x(u^{(i)}, v^{(i)}) = u^{(i)}(x)(pv^{(i)'})'(x) - (pu^{(i)'})'(x)v^{(i)}(x)$ ($x \in I_i, i=1, 2$). Then the equality $W_x^{(1)} = \rho W_x^{(2)}$ holds. For convenience, we denote $W_x := W_x^{(1)} = \rho W_x^{(2)}$. Since the wronskian of any two solutions of Equation (1.3) is constant, we have $W_x(u, v) = 1$. Then, u and v are linearly independent and they form a fundamental system of solutions of equation (1.3). By the condition A1, we get $u, v \in L^2(I)$ and moreover, $u, v \in \mathcal{D}$. So, the values $[y, u]_\infty$ and $[y, v]_\infty$ exist and are finite for every $y \in \mathcal{D}$. By using Green's formula (1.2) and the conditions (1.5)-(1.6), we can get

$$(1.7) \quad \begin{aligned} [y, u]_\infty &= y(a) + \int_a^\infty u(x) \overline{l(y(x))} dx, \\ [y, v]_\infty &= (py')(a) + \int_a^\infty v(x) \overline{l(y(x))} dx. \end{aligned}$$

Now, we will add to problem (1.1) the boundary conditions

$$(1.8) \quad \begin{aligned} y(a) \cos \alpha + (py')(a) \sin \alpha &= d_1, \\ [y, u]_\infty \cos \beta + [y, v]_\infty \sin \beta &= d_2, \end{aligned}$$

and impulsive conditions

$$(1.9) \quad Y(c+) = CY(c-), \quad Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad \det C = \rho > 0,$$

where $\alpha, \beta \in \mathbb{R}$, and d_1, d_2 are arbitrary given real numbers, and

$$(A3) \quad \omega := \cos \alpha \sin \beta - \cos \beta \sin \alpha \neq 0.$$

Since the function y in (1.8) satisfies Equation (1.1), we have

$$\begin{aligned} [y, u]_\infty &= y(a) + \int_a^\infty u(x) f(x, y(x)) dx, \\ [y, v]_\infty &= (py')(a) + \int_a^\infty v(x) f(x, y(x)) dx. \end{aligned}$$

2. Green's function

In this section, we construct an appropriate Green's function. So, we will reduce the boundary-value problem (1.1), (1.8), (1.9) to a fixed point problem.

Let us consider the linear boundary value problem

$$(2.1) \quad -(p(x)y')' + q(x)y = h(x), \quad x \in I, \quad h \in H,$$

$$(2.2) \quad \left. \begin{aligned} y(a) \cos \alpha + (py')(a) \sin \alpha &= 0, \\ [y, u]_{\infty} \cos \beta + [y, v]_{\infty} \sin \beta &= 0, \quad \alpha, \beta \in \mathbb{R}, \\ Y(c+) &= CY(c-), \quad Y := \begin{pmatrix} y \\ py' \end{pmatrix}, \quad \det C = \rho > 0, \end{aligned} \right\}$$

where y is a desired solution, u and v are solutions to the equation (1.3) under the conditions (1.5)-(1.6).

Define

$$(2.3) \quad \varphi(x) = \cos \alpha u(x) + \sin \alpha v(x), \quad \psi(x) = \cos \beta u(x) + \sin \beta v(x),$$

where $W_x(\varphi, \psi) = \omega$. It is clear that these functions are solutions to the equation (1.3) and are in H . Further, we have

$$\begin{aligned} [\varphi, u]_x &= \varphi(a) = -\sin \alpha, \quad [\varphi, v]_x = (p\varphi)'(a) = \cos \alpha, \quad (x \in I_1), \\ [\psi, u]_x &= \psi(a) = -\sin \beta, \quad [\psi, v]_x = (p\psi)'(a) = \cos \beta, \quad (x \in I_1), \\ [\psi, u]_{\infty} &= -\rho \sin \beta, \quad [\psi, v]_{\infty} = \rho \cos \beta, \\ \Phi(c+) &= C\Phi(c-), \quad \Phi(x) := \begin{pmatrix} \varphi(x) \\ (p\varphi)'(x) \end{pmatrix}, \\ \Psi(c+) &= C\Psi(c-), \quad \Psi(x) := \begin{pmatrix} \psi(x) \\ (p\psi)'(x) \end{pmatrix}. \end{aligned}$$

Let us introduce the function

$$(2.4) \quad G(x, t) = \begin{cases} \frac{\varphi(x)\psi(t)}{\omega}, & \text{if } a \leq x \leq t < \infty, \quad x \neq c, \quad t \neq c, \\ \frac{\varphi(t)\psi(x)}{\omega}, & \text{if } a \leq t \leq x < \infty, \quad x \neq c, \quad t \neq c. \end{cases}$$

$G(x, t)$ is called the Green's function of the boundary-value problem (2.1)-(2.2). Since $\varphi, \psi \in H$, we have

$$(2.5) \quad \int_a^{\infty} \int_a^{\infty} |G(x, t)|^2 dx dt < \infty,$$

i.e., $G(x, t)$ is a Hilbert-Schmidt kernel.

Theorem 2.1. *The function*

$$(2.6) \quad y(x) = \int_a^c G(x, t) h(t) dt + \gamma \int_c^{\infty} G(x, t) h(t) dt, \quad x \in I,$$

is the solution of the boundary-value problem (2.1)-(2.2).

Proof. By a variation of constants formula, the general solution of the equation (2.1) has the form

$$(2.7) \quad y(x) = \begin{cases} k_1 \varphi^{(1)}(x) + k_2 \psi^{(1)}(x) \\ + \frac{\psi^{(1)}(x)}{\omega} \int_a^x \varphi^{(1)}(t) h(t) dt \\ + \frac{\varphi^{(1)}(x)}{\omega} \int_x^c \psi^{(1)}(t) h(t) dt, & x \in I_1, \\ k_3 \varphi^{(2)}(x) + k_4 \psi^{(2)}(x) \\ + \frac{\varphi^{(2)}(x)}{\omega} \int_c^x \psi^{(2)}(t) h(t) dt \\ + \frac{\psi^{(2)}(x)}{\omega} \int_x^\infty \varphi^{(2)}(t) h(t) dt, & x \in I_2, \end{cases}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants.

By (2.7), we get

$$(py)'(x) = \begin{cases} k_1 (p\varphi^{(1)})'(x) + k_2 (p\psi^{(1)})'(x) \\ + \frac{(p\psi^{(1)})'(x)}{\omega} \int_a^x \varphi^{(1)}(t) h(t) dt \\ + \frac{(p\varphi^{(1)})'(x)}{\omega} \int_x^c \psi^{(1)}(t) h(t) dt, & x \in I_1, \\ k_3 (p\varphi^{(2)})'(x) + k_4 (p\psi^{(2)})'(x) \\ + \frac{\varphi^{(2)}(x)}{\omega} (p\psi^{(2)})'(x) \int_c^x \varphi^{(2)}(t) h(t) dt \\ + \frac{\psi^{(2)}(x)}{\omega} (p\varphi^{(2)})'(x) \int_x^\infty \psi^{(2)}(t) h(t) dt, & x \in I_2. \end{cases}$$

Hence, we have

$$\begin{aligned} y(a) &= k_1 \varphi^{(1)}(a) + k_2 \psi^{(1)}(a) + \frac{\varphi^{(1)}(a)}{\omega} \int_a^c \psi^{(1)}(t) h(t) dt \\ &= -k_1 \sin \alpha - k_2 \sin \beta - \frac{1}{\omega} \sin \alpha \int_a^c \varphi^{(1)}(t) h(t) dt, \\ (2.8) \quad (py)'(a) &= k_1 (p\varphi^{(1)})'(a) + k_2 (p\psi^{(1)})'(a) \\ &\quad + \frac{1}{\omega} (p\varphi^{(1)})'(a) \int_a^c \psi^{(1)}(t) h(t) dt \\ &= k_1 \cos \alpha + k_2 \cos \beta + \frac{1}{\omega} \cos \alpha \int_a^c \varphi^{(1)}(t) h(t) dt. \end{aligned}$$

Substituting (2.8) into (2.2), we get

$$k_2 (\cos \alpha \sin \beta - \sin \alpha \cos \beta) = 0, \quad k_2 \omega = 0,$$

i.e., $k_2 = 0$. Further, we have

$$[y, u]_x = y(x)(pu')(x) - (py')(x)u(x)$$

$$= \begin{cases} k_1[\varphi^{(1)}, u]_x + \frac{1}{\omega}[[\psi^{(1)}(x), u]_x \int_a^x \varphi^{(1)}(t) h(t) dt \\ + \frac{1}{\omega}[\varphi^{(1)}(x), u]_x \int_x^c \psi^{(1)}(t) h(t) dt, & x \in I_1, \\ k_3[\varphi^{(2)}, u]_x + k_4[\psi^{(2)}, u]_x \\ + \frac{\gamma}{\omega}[\psi^{(2)}, u]_x \int_c^x \varphi^{(2)}(t) h(t) dt \\ + \frac{\gamma}{\omega}[\varphi^{(2)}, u]_x \int_x^\infty \psi^{(2)}(t) h(t) dt, & x \in I_2. \end{cases}$$

Thus

$$[y, u]_\infty = -k_3\rho \sin \alpha - k_4\rho \sin \beta - \frac{\gamma}{\omega}\rho \sin \beta \int_c^\infty \varphi^{(2)}(t) h(t) dt.$$

Similarly, we get

$$[y, v]_x = y(x)(pv')(x) - (py')(x)v(x)$$

$$= \begin{cases} k_1[\varphi^{(1)}, v]_x \\ + \frac{1}{\omega}[[\psi^{(1)}(x), v]_x \int_a^x \varphi^{(1)}(t) h(t) dt \\ + \frac{1}{\omega}[\varphi^{(1)}(x), v]_x \int_x^c \psi^{(1)}(t) h(t) dt, & x \in I_1, \\ k_3[\varphi^{(2)}, v]_x + k_4[\psi^{(2)}, v]_x \\ + \frac{\gamma}{\omega}[\psi^{(2)}, v]_x \int_c^x \varphi^{(2)}(t) h(t) dt \\ + \frac{\gamma}{\omega}[\varphi^{(2)}, v]_x \int_x^\infty \psi^{(2)}(t) h(t) dt, & x \in I_2, \end{cases}$$

and

$$[y, v]_\infty = k_3\rho \cos \alpha + k_4\rho \cos \beta + \frac{\gamma}{\omega}\rho \cos \beta \int_c^\infty \varphi^{(2)}(t) h(t) dt.$$

From the conditions (2.2), we obtain

$$k_3(\sin \alpha \cos \beta - \cos \alpha \sin \beta) = 0.$$

Hence, $k_3 = 0$. Similarly, we have

$$\begin{aligned}
 Y(c+) &= \begin{pmatrix} y(c+) \\ (py')(c+) \end{pmatrix} = \begin{pmatrix} k_4 \psi^{(2)}(c+) \\ k_4 (p\psi^{(2)})'(c+) \end{pmatrix} \\
 &+ \begin{pmatrix} \frac{\gamma}{\omega} \varphi^{(2)}(c+) \int_c^\infty \psi^{(2)}(t) h(t) dt \\ \frac{\gamma}{\omega} (p\varphi^{(2)})'(c+) \int_c^\infty \psi^{(2)}(t) h(t) dt \end{pmatrix} \\
 &= k_4 \begin{pmatrix} \psi^{(2)}(c+) \\ (p\psi^{(2)})'(c+) \end{pmatrix} \\
 &+ \frac{\gamma}{\omega} \int_c^\infty \psi^{(2)}(t) h(t) dt \begin{pmatrix} \varphi^{(2)}(c+) \\ (p\varphi^{(2)})'(c+) \end{pmatrix} \\
 &= k_4 \Psi(c+) + \left\{ \frac{\gamma}{\omega} \int_c^\infty \psi^{(2)}(t) h(t) dt \right\} \Phi(c+)
 \end{aligned}$$

and

$$\begin{aligned}
 Y(c-) &= \begin{pmatrix} y(c-) \\ (py')(c-) \end{pmatrix} \\
 &= \begin{pmatrix} k_1 \varphi^{(1)}(c-) \\ k_1 (p\varphi^{(1)})'(c-) \end{pmatrix} + \begin{pmatrix} \frac{\psi^{(1)}(c-)}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \\ \frac{(p\psi^{(1)})'(c-)}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \end{pmatrix} \\
 &= k_1 \begin{pmatrix} \varphi^{(1)}(c-) \\ (p\varphi^{(1)})'(c-) \end{pmatrix} + \frac{1}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \begin{pmatrix} \psi^{(1)}(c-) \\ (p\psi^{(1)})'(c-) \end{pmatrix} \\
 &= k_1 \Phi(c-) + \left\{ \frac{1}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \right\} \Psi(c-).
 \end{aligned}$$

By the conditions (2.2), we obtain

$$\begin{aligned}
 &k_4 \Psi(c+) + \left\{ \frac{\gamma}{\omega} \int_c^\infty \psi^{(2)}(t) h(t) dt \right\} \Phi(c+) \\
 &= C \left\{ k_1 \Phi(c-) + \left\{ \frac{1}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \right\} \Psi(c-) \right\}.
 \end{aligned}$$

Using the conditions (2.) and (2.), we get

$$\begin{aligned}
 & \Phi(c-)\left\{\frac{\gamma}{\omega}\int_c^\infty\psi^{(2)}(t)h(t)dt-k_1\right\} \\
 &= \Psi(c-)\left\{\frac{1}{\omega}\int_a^c\varphi^{(1)}(t)h(t)dt-k_4\right\}, \\
 &\left(\begin{array}{c}\varphi^{(1)}(c-)\\(p\varphi^{(1)'}) (c-)\end{array}\right)\left\{\frac{\gamma}{\omega}\int_c^\infty\psi^{(2)}(t)h(t)dt-k_1\right\} \\
 &= \left(\begin{array}{c}\psi^{(1)}(c-)\\(p\psi^{(1)'}) (c-)\end{array}\right)\left\{\frac{1}{\omega}\int_a^c\varphi^{(1)}(t)h(t)dt-k_4\right\}.
 \end{aligned}$$

So, we have the following linear equation system

$$\begin{aligned}
 & k_4\psi^{(1)}(c-)-k_1\varphi^{(1)}(c-) \\
 &= \left\{\frac{1}{\omega}\int_a^c\varphi^{(1)}(t)h(t)dt\right\}\psi^{(1)}(c-) \\
 &- \left\{\frac{\gamma}{\omega}\int_c^\infty\psi^{(2)}(t)h(t)dt\right\}\varphi^{(1)}(c-), \\
 & k_4(p\psi^{(1)'}) (c-)-k_1(p\varphi^{(1)'}) (c-) \\
 &= \left\{\frac{1}{\omega}\int_a^c\varphi^{(1)}(t)h(t)dt\right\}(p\psi^{(1)'}) (c-) \\
 &- \left\{\frac{\gamma}{\omega}\int_c^\infty\psi^{(2)}(t)h(t)dt\right\}(p\varphi^{(1)'}) (c-),
 \end{aligned}$$

i.e.,

$$\begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)})'(c-) & (p\varphi^{(1)})'(c-) \end{pmatrix} \begin{pmatrix} k_4 \\ -k_1 \end{pmatrix} \\ = \begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)})'(c-) & (p\varphi^{(1)})'(c-) \end{pmatrix} \\ \times \begin{pmatrix} \frac{1}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \\ -\frac{\gamma}{\omega} \int_c^\infty \psi^{(2)}(t) h(t) dt \end{pmatrix}.$$

Hence, we have the following determinant of this linear equation system

$$\begin{vmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)})'(c-) & (p\varphi^{(1)})'(c-) \end{vmatrix} = -\omega.$$

Since this determinant is different from zero, the solution of this system is unique. If we solve this system, we have the following equalities

$$k_1 = \frac{\gamma}{\omega} \int_c^\infty \psi^{(2)}(t) h(t) dt, k_4 = \frac{1}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt.$$

From what has already been done, we have

$$y(x) = \begin{cases} \varphi^{(1)}(x) \frac{\gamma}{\omega} \int_c^\infty \psi^{(2)}(t) h(t) dt \\ + \frac{\psi^{(1)}(x)}{\omega} \int_a^x \varphi^{(1)}(t) h(t) dt \\ + \frac{\varphi^{(1)}(x)}{\omega} \int_x^c \psi^{(1)}(t) h(t) dt, & x \in I_1, \\ \psi^{(2)}(x) \frac{1}{\omega} \int_a^c \varphi^{(1)}(t) h(t) dt \\ + \frac{\gamma}{\omega} \psi^{(2)}(x) \int_c^x \varphi^{(2)}(t) h(t) dt \\ + \frac{\gamma}{\omega} \varphi^{(2)}(x) \int_x^\infty \psi^{(2)}(t) h(t) dt, & x \in I_2, \end{cases}$$

i.e., (2.4) and (2.6) hold. \square

Thus we have a

Theorem 2.2. *The unique solution to the equation (2.1) under the conditions (1.8)-(1.9) is given by the formula*

$$y(x) = w(x) + \langle G(x, \cdot), \overline{h(\cdot)} \rangle,$$

where

$$w(x) = \frac{d_1}{\omega} \varphi(x) - \frac{d_2}{\omega} \psi(x).$$

Proof. By the conditions (2.)-(2.), the function $w(x)$ is a unique solution of the equation (1.3) satisfying the conditions (1.8)-(1.9). By Theorem 1 the function $\langle G(x, \cdot), \overline{h(\cdot)} \rangle$ a unique solution to the equation (2.1) satisfying the conditions (2.2). This finishes the proof. \square

From Theorem 2, the boundary-value problem (1.1), (1.8), (1.9) in H is equivalent to the non-linear integral equation

$$(2.9) \quad y(x) = w(x) + \langle G(x, \cdot), f(\cdot, y(\cdot)) \rangle, \quad x \in I,$$

where the functions $w(x)$ and $G(x, t)$ are defined above. Hence, we shall study the equation (2.9).

By (1.4) and (2.5), we can define the operator $T : H \rightarrow H$ by the formula

$$(2.10) \quad (Ty)(x) = w(x) + \langle G(x, \cdot), f(\cdot, y(\cdot)) \rangle, \quad x \in I,$$

where $y, w \in H$. Then the equation (2.9) can be written as $y = Ty$.

Now, we search the fixed points of the operator T because it is equivalent to solving the equation (2.9).

3. The fixed points of the operator T

In this section, we investigate the fixed points of the operator T by using the following Banach fixed point theorem:

Definition 3.1. [[16]] Let A be a mapping of a metric space R into itself. Then x is called a fixed point of A if $Ax = x$. Suppose there exists a number $\alpha < 1$ such that

$$\rho(Ax, Ay) \leq \alpha \rho(x, y)$$

for every pair of points $x, y \in R$. Then A is said to be a contraction mapping.

Theorem 3.1. [16] *Every contraction mapping A defined on a complete metric space R has a unique fixed point.*

Theorem 3.2. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. Further, let the function $f(x, y)$ satisfy the following Lipschitz condition: there*

exists a constant $K > 0$ such that

$$\begin{aligned} & \int_a^c \left| f^{(1)}(x, y^{(1)}(x)) - f^{(1)}(x, z^{(1)}(x)) \right|^2 dx \\ & + \gamma \int_c^\infty \left| f^{(2)}(x, y^{(2)}(x)) - f^{(2)}(x, z^{(2)}(x)) \right|^2 dx \\ & \leq K^2 \left(\int_a^c \left| y^{(1)}(x) - z^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) - z^{(2)}(x) \right|^2 dx \right) \\ & = K^2 \|y - z\|^2 \end{aligned}$$

for all $y, z \in H$. If

$$(3.1) \quad K \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right) < 1,$$

then the boundary-value problem (1.1), (1.8), (1.9) has a unique solution in H .

Proof. It suffices to prove that the operator T is a contraction operator. For $y, z \in H$, we have

$$\begin{aligned} |Ty(x) - Tz(x)|^2 &= |\langle G(x, \cdot), [f(\cdot, y(\cdot)) - f(\cdot, z(\cdot))] \rangle|^2 \\ &\leq \|G(x, \cdot)\|^2 \|f(\cdot, y(\cdot)) - f(\cdot, z(\cdot))\|^2 \\ &\leq K^2 \|G(x, \cdot)\|^2 \|y - z\|^2, \quad x \in I. \end{aligned}$$

Thus, we get

$$\|Ty - Tz\| \leq \alpha \|y - z\|,$$

where

$$\alpha = K \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right)^{\frac{1}{2}} < 1,$$

i.e., T is a contraction mapping. \square

Now, our next claim is that the function $f(x, y)$ satisfies a Lipschitz condition on a subset of H but not of the whole space.

Theorem 3.3. Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let the function $f(x, y)$ satisfy the following Lipschitz condition: there

exist constants $M, K > 0$ such that

$$\begin{aligned}
& \int_a^c \left| f^{(1)}(x, y^{(1)}(x)) - f^{(1)}(x, z^{(1)}(x)) \right|^2 dx \\
& + \gamma \int_c^\infty \left| f^{(2)}(x, y^{(2)}(x)) - f^{(2)}(x, z^{(2)}(x)) \right|^2 dx \\
& \leq K^2 \left(\int_a^c \left| y^{(1)}(x) - z^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) - z^{(2)}(x) \right|^2 dx \right) \\
& = K^2 \|y - z\|^2
\end{aligned}$$

for all y and z in $S_M = \{t \in H : \|t\| \leq M\}$, where K may depend on M . If

$$\begin{aligned}
& \left\{ \int_a^c \left| w^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| w^{(2)}(x) \right|^2 dx \right\}^{1/2} \\
& + \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right)^{\frac{1}{2}} \\
& \times \sup_{y \in S_M} \left\{ \begin{aligned} & \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) - f^{(1)}(t, z^{(1)}(t)) \right|^2 dt \\ & + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) - f^{(2)}(t, z^{(2)}(t)) \right|^2 dt \end{aligned} \right\}^{1/2} \\
& \leq M
\end{aligned}$$

and

$$(3.2) \quad K \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right)^{\frac{1}{2}} < 1,$$

then the boundary-value problem (1.1), (1.8), (1.9) has a unique solution with

$$\int_a^c \left| y^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) \right|^2 dx \leq M^2.$$

Proof. It is clear that S_M is a closed set of H . We first prove that the operator T

maps S_M into itself. For $y \in S_M$ we have

$$\begin{aligned} \|Ty\| &= \|w(x) + \langle G(x, \cdot), f(\cdot, y(\cdot)) \rangle\| \leq \|w\| + \|\langle G(x, \cdot), f(\cdot, y(\cdot)) \rangle\| \\ &\leq \|w\| + \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right)^{\frac{1}{2}} \\ &\quad \times \sup_{y \in S_M} \left\{ \begin{aligned} &\int_a^c |f^{(1)}(t, y^{(1)}(t)) - f^{(1)}(t, z^{(1)}(t))|^2 dt \\ &+ \gamma \int_c^\infty |f^{(2)}(t, y^{(2)}(t)) - f^{(2)}(t, z^{(2)}(t))|^2 dt \end{aligned} \right\}^{1/2} \leq M. \end{aligned}$$

Consequently, $T : S_M \rightarrow S_M$.

We can now proceed analogously to the proof of Theorem 5. So, we can get

$$\|Ty - Tz\| \leq \alpha \|y - z\|, \quad y, z \in S_M.$$

If we apply the Banach fixed point theorem, then we obtain a unique solution of the boundary-value problem (1.1), (1.8), (1.9) in S_M . \square

4. An existence theorem without uniqueness

In this section, we get an existence theorem without uniqueness of solution. Therefore, we will use the following Schauder fixed point theorem:

Definition 4.1. [[11]] An operator acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Theorem 4.1. [11] Let \mathbf{B} be a Banach space and \mathbf{S} a nonempty bounded, convex, and closed subset of \mathbf{B} . Assume $A : \mathbf{B} \rightarrow \mathbf{B}$ is a completely continuous operator. If the operator A leaves the set \mathbf{S} invariant, i.e., if $A(\mathbf{S}) \subset \mathbf{S}$, then A has at least one fixed point in \mathbf{S} .

A set $S \subset H$ is relatively compact iff S is bounded and for every $\varepsilon > 0$ (i) there exists $\delta > 0$ such that $\|y(x+h) - y(x)\| < \varepsilon$ for all $y \in S$ and all $h \geq 0$ with $h < \delta$, (ii) there exists a number $N > 0$ such that $\int_N^\infty |y(x)|^2 dx < \varepsilon$ for all $y \in S$ ([11]).

Now, we give

Theorem 4.2. The operator T defined by (2.10) is completely continuous operator under the conditions (A1), (A2) and (A3).

Proof. Let $y_0 \in H$. Then, we have

$$\begin{aligned} & |(Ty)(x) - (Ty_0)(x)|^2 \\ &= |\langle G(x, \cdot), [f(\cdot, y(\cdot)) - f(\cdot, y_0(\cdot))] \rangle|^2 \\ &\leq \|G(x, \cdot)\|^2 \left\{ \begin{aligned} & \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) - f^{(1)}(t, y_0^{(1)}(t)) \right|^2 dt \\ & + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) - f^{(2)}(t, y_0^{(2)}(t)) \right|^2 dt \end{aligned} \right\}^2. \end{aligned}$$

Thus

$$\begin{aligned} & \|Ty - Ty_0\|^2 \\ &\leq K \left\{ \begin{aligned} & \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) - f^{(1)}(t, y_0^{(1)}(t)) \right|^2 dt \\ & + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) - f^{(2)}(t, y_0^{(2)}(t)) \right|^2 dt \end{aligned} \right\}, \end{aligned}$$

where

$$K = \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right).$$

We know that an operator F defined by $Fy(x) = f(x, y(x))$ is continuous in H under the condition (A2) (see [17]). Hence, for a given $\epsilon > 0$, we can find a $\delta > 0$ such that $\|y - y_0\| < \delta$ implies

$$\left\{ \begin{aligned} & \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) - f^{(1)}(t, y_0^{(1)}(t)) \right|^2 dt \\ & + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) - f^{(2)}(t, y_0^{(2)}(t)) \right|^2 dt \end{aligned} \right\} < \frac{\epsilon^2}{K^2}.$$

From (4.), we get

$$\|Ty - Ty_0\| < \epsilon,$$

i.e., T is continuous.

Set $Y = \{y \in H : \|y\| \leq m\}$. By (3.3), we have

$$\|Ty\| \leq \|w\| + \left\{ \begin{aligned} & K \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) \right|^2 dt \\ & + \gamma K \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) \right|^2 dt \end{aligned} \right\}^{1/2}, \text{ for all } y \in Y.$$

Furthermore, using (1.4), we get

$$\begin{aligned}
& \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) \right|^2 dt + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) \right|^2 dt \\
& \leq \int_a^c \left[g^{(1)}(t) + \vartheta \left| y^{(1)}(t) \right| \right]^2 dt + \gamma \int_c^\infty \left[g^{(2)}(t) + \vartheta \left| y^{(2)}(t) \right| \right]^2 dt \\
& \leq 2 \int_a^c \left[\left(g^{(1)} \right)^2(t) + \vartheta^2 \left| y^{(1)}(t) \right|^2 \right] dt \\
& \quad + 2\gamma \int_c^\infty \left[\left(g^{(2)} \right)^2(t) + \vartheta^2 \left| y^{(2)}(t) \right|^2 \right] dt \\
& = 2(\|g\|^2 + \vartheta^2 \|y\|^2) \leq 2(\|g\|^2 + \vartheta^2 m^2).
\end{aligned}$$

Thus, for all $y \in Y$, we obtain

$$\|Ty\| \leq \|w\| + \left[2K \left(\|g\|^2 + \vartheta^2 m \right) \right]^{1/2},$$

i.e., $T(Y)$ is a bounded set in H .

Moreover, for all $y \in Y$, we have

$$\begin{aligned}
& \int_a^c \left| (Ty^{(1)})(x+h) - (Ty^{(1)})(x) \right|^2 dx \\
& + \gamma \int_c^\infty \left| (Ty^{(2)})(x+h) - (Ty^{(2)})(x) \right|^2 dx \\
& = \left\| \langle [G(x+h, \cdot) - G(x, \cdot)], f(\cdot, y(\cdot)) \rangle \right\|^2 \\
& \leq \left(\int_a^c \int_a^c |G(x+h, t) - G(x, t)|^2 dx dt \right. \\
& \quad \left. + \gamma \int_c^\infty \int_c^\infty |G(x+h, t) - G(x, t)|^2 dx dt \right) \\
& \quad \times \left\{ \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) \right|^2 dt \right. \\
& \quad \left. + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) \right|^2 dt \right\}^2 \\
& \leq 2 \left(\|g\|^2 + \vartheta^2 m \right) \left(\int_a^c \int_a^c |G(x+h, t) - G(x, t)|^2 dx dt \right. \\
& \quad \left. + \gamma \int_c^\infty \int_c^\infty |G(x+h, t) - G(x, t)|^2 dx dt \right)
\end{aligned}$$

From (2.5), there exists a $\delta > 0$ such that

$$\begin{aligned} & \int_a^c \left| Ty^{(1)}(x+h) - Ty^{(1)}(x) \right|^2 dx \\ & + \gamma \int_c^\infty \left| Ty^{(2)}(x+h) - Ty^{(2)}(x) \right|^2 dx < \epsilon^2, \end{aligned}$$

for given $\epsilon > 0$, all $y \in Y$ and all $h < \delta$.

Further, for all $y \in Y$, we have ($N > c$)

$$\begin{aligned} & \int_N^\infty \left| (Ty^{(2)})(x) \right|^2 dx \\ & \leq \int_N^\infty \left| w^{(2)}(x) \right|^2 dx + 2 \left(\|g\|^2 + v^2 m \right) \int_N^\infty \|G(x, \cdot)\|^2 dx. \end{aligned}$$

So, from (2.5), we see that for a given $\epsilon > 0$ there exists a positive number N , depending only on ϵ such that

$$\int_N^\infty \left| (Ty^{(2)})(x) \right|^2 dx < \epsilon^2,$$

for all $y \in Y$.

Thus $T(Y)$ is a relatively compact in H , i.e., the operator T is completely continuous. \square

Theorem 4.3. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let there exist constants $M > 0$ such that*

$$\begin{aligned} & \left\{ \int_a^c \left| w^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| w^{(2)}(x) \right|^2 dx \right\}^{1/2} \\ & + \left(\int_a^c \int_a^c |G(x, t)|^2 dx dt + \gamma \int_c^\infty \int_c^\infty |G(x, t)|^2 dx dt \right) \\ & \times \sup_{y \in S_M} \left\{ \begin{aligned} & \int_a^c \left| f^{(1)}(t, y^{(1)}(t)) - f^{(1)}(t, z^{(1)}(t)) \right|^2 dt \\ & + \gamma \int_c^\infty \left| f^{(2)}(t, y^{(2)}(t)) - f^{(2)}(t, z^{(2)}(t)) \right|^2 dt \end{aligned} \right\}^{1/2} \\ & \leq M, \end{aligned}$$

where $S_M = \{y \in H : \|y\| \leq M\}$. Then the boundary-value problem (1.1), (1.8), (1.9) has at least one solution with

$$\int_a^c \left| y^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) \right|^2 dx \leq M^2.$$

Proof. Let us define an operator $T : H \rightarrow H$ by (2.10). From theorems 6, 9 and (4.3), we conclude that T maps the set S_M into itself. It is clear that the set S_M is bounded, convex and closed. Using Theorem 8, the theorem follows. \square

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